Relative monads and their many guises

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1970 : Devices

In 1970, Walters introduced the notion of device to capture classical universal algebra, and which was shown to generalise the notion of monad.

However, the concept was overlooked, and devices were forgotten...

2010: Relative monads

Forty years later, the concept of relative monad was introduced by Altenkirch, Chapman, and Uustalu as a generalisation of the concept of monad, from a structured endofunctor to an arbitrary functor with structure.

Crucially, the authors related relative monads to the relative adjunctions of Ulmer.

Relative monads and devices

In fact, the definition of relative monad is reminiscent to that of device, and the two turn out to be equivalent.

Relative monads in extension form

[ACU10]

Fix a functor $j : A \to E$.

A j-relative monad comprises

- a functor $t : A \to E$
- a natural transformation $\eta : j \Rightarrow t$
- a natural transformation

$$\dagger : E(j=, t-) \Rightarrow E(t=, t-)$$

satisfying unit and associativity laws.

Where does the concept of relative monad come from, categorically?

Monads as monoids

For any category A, the category $\mathrm{Cat}(A,A)$ of endofunctors on A is equipped with strict monoidal structure given by functor composition.

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Can this be generalised to arbitrary functor categories?

A hint of a solution

ACU show that, assuming the existence of enough (pointwise) extensions along $A \xrightarrow{j} E$, $Cat(A, E)$ may be equipped with skew-monoidal structure, in which the unit is $A \xrightarrow{j} E$, and the tensor of $A \xrightarrow{f} E$ and $A \xrightarrow{g} E$ is given by left extension:

$$\begin{tikzcd}
A \arrow[r, "f"] & E \arrow[dr, dashed, "j \triangleright g"] & \\
& A \arrow[u, "g"] \arrow[r, "j"'] & E
\end{tikzcd}$$

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Furthermore, monoids in this skew-monoidal category of functors are precisely j-relative monads.

However, the assumption of enough left extensions along j is dissatisfying; after all, monads are always monoids in a category of functors.

$\mathcal{V}$-distributors (AKA $\mathcal{V}$-profunctors)

Let $\mathcal{V}$ be a monoidal category. A $\mathcal{V}$-distributor $A \overset{p}{\nrightarrow} B$ comprises

$$p(b,a) \in \mathcal{V} \qquad (b \in B,\ a \in A)$$

functorial contravariantly in b and covariantly in a.

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$\mathcal{V}$-distributors $A \overset{p}{\nrightarrow} B \overset{q}{\nrightarrow} C$ cannot generally be composed. We will denote by $q \odot p$ the composite when it exists.

[SD97]

$\mathcal{V}$-forms

Consider $\mathcal{V}$-distributors

$$\begin{tikzcd} A_n \arrow[r, "p_n", "\shortmid" marking] \arrow[rrrr, bend right, "q"', "\shortmid" marking] & A_{n-1} \arrow[r, "p_{n-1}", "\shortmid" marking] & \cdots \arrow[r, "p_2", "\shortmid" marking] & A_1 \arrow[r, "p_1", "\shortmid" marking] & A_0 \end{tikzcd}$$

A $\mathcal{V}$-form $p_1, \ldots, p_n \Rightarrow q$ comprises a family

$$\left\{ p_1(a_0, a_1) \otimes \cdots \otimes p_n(a_{n-1}, a_n) \to q(a_0, a_n) \right\}_{a_i \in A_i}$$

of morphisms in $\mathcal{V}$, satisfying naturality laws.

Multicategories of endodistributors

For each $\mathcal{V}$-category A, there is a multicategory $\mathcal{V}\text{-}\mathrm{Dist}[A,A]$ whose objects are $\mathcal{V}$-distributors $A \to\!\!\!\!\!\!+\; A$ and whose multimorphisms are $\mathcal{V}$-forms

$$\begin{tikzcd} A \arrow[r, "p_n", "\shortmid" marking] \arrow[rrrrr, bend right, "q"', "\shortmid" marking] & A \arrow[r, "p_{n-1}", "\shortmid" marking] & \cdots \arrow[d, phantom, "\Downarrow"] \arrow[r, "p_2", "\shortmid" marking] & A \arrow[r, "p_1", "\shortmid" marking] & A \\ & & {} & & \end{tikzcd}$$

Denote by $\mathcal{V}\text{-}\mathrm{Dist}(A,A)$ the underlying category.

Representable distributors

Let $A \xrightarrow{f} B$ be a $\mathcal{V}$-functor. There is an induced representable $\mathcal{V}$-distributor

$$B(1,f) : A \nrightarrow B \qquad (b,a) \mapsto B(b,fa)$$

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and an induced corepresentable $\mathcal{V}$-distributor

$$B(f,1) : B \nrightarrow A \qquad (a,b) \mapsto B(fa,b)$$

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A $\mathcal{V}$-form $B(1,f) \Rightarrow B(1,g)$ is just a $\mathcal{V}$-natural transformation $f \Rightarrow g$.

Multicategories of endofunctors

There is a submulticategory $\mathcal{V}\text{-}\mathrm{Cat}[A,A]$ of $\mathcal{V}\text{-}\mathrm{Dist}[A,A]$ formed by the representable $\mathcal{V}$-distributors.

We may always compose representable $\mathcal{V}$-distributors, and so $\mathcal{V}\text{-}\mathrm{Cat}[A,A]$ is representable: i.e. a monoidal category. It is equivalent to the usual strict monoidal category $\mathcal{V}\text{-}\mathrm{Cat}(A,A)$ of $\mathcal{V}$-endofunctors on A.

Skew composition I

Let $\mathcal{A}$ and $\mathcal{E}$ be categories. Given functors

$$\mathcal{A} \xrightarrow{f} \mathcal{E} \qquad \mathcal{A} \xrightarrow{g} \mathcal{E}$$

we cannot in general form any sort of composite $\mathcal{A} \xrightarrow{g \cdot f} \mathcal{E}$, since the codomain of f does not match the domain of g.

Skew composition I

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However, suppose we had a fixed functor $E \xrightarrow{j^*} A$. Then we could form a composite

$$A \xrightarrow{f} E \xrightarrow{j^*} A \xrightarrow{g} E$$

Skew composition II

Fix a $\mathcal{V}$-functor $A \xrightarrow{j} E$. Given $\mathcal{V}$-functors

$$A \xrightarrow{f} E \qquad A \xrightarrow{g} E$$

we may form a chain

$$A \xrightarrow[\quad +\quad]{E(1,f)} E \xrightarrow[\quad +\quad]{E(j,1)} A \xrightarrow[\quad +\quad]{E(1,g)} E$$

by considering f, g as representable $\mathcal{V}$-distributors, and using the corepresentable $E(j,1)$ to facilitate composition.

Skew-multicategories of distributors

Theorem. Let $j : A \to E$ be a $\mathcal{V}$-functor. There is a skew-multicategory $\mathcal{V}\text{-}\mathrm{Dist}[j]$ whose objects are $\mathcal{V}$-distributors $A \nrightarrow E$ and whose multimorphisms are $\mathcal{V}$-forms

$$\begin{tikzcd} A \arrow[r, "p_n", "\shortmid"{marking}] \arrow[rrrrrr, bend right, "\shortmid"{marking}, "q"', ""{name=Q}] & E \arrow[r, "E(j,1)", "\shortmid"{marking}] & A \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\shortmid"{marking}] & E \arrow[r, "E(j,1)", "\shortmid"{marking}] & A \arrow[r, "p_1", "\shortmid"{marking}] & E \arrow[from=1-4, to=Q, Rightarrow, shorten=2mm] \end{tikzcd}$$

Skew-multicategories of functors

There is a sub skew-multicategory $\mathcal{V}\text{-}\mathbf{Cat}[j]$ of $\mathcal{V}\text{-}\mathbf{Dist}[j]$ formed by the representables.

$$\begin{tikzcd}
A \arrow[r, "{E(1,f_n)}", "\shortmid"{marking}] \arrow[rrrrrr, bend right=20, "{E(1,g)}"', "\shortmid"{marking}] & E \arrow[r, "{E(j,1)}", "\shortmid"{marking}] & A \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & E \arrow[r, "{E(j,1)}", "\shortmid"{marking}] & A \arrow[r, "{E(1,f_1)}", "\shortmid"{marking}] & E \\
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\end{tikzcd}$$

We cannot generally compose representable $\mathcal{V}$-distributors with corepresentable ones, so $\mathcal{V}\text{-}\mathrm{Cat}[j]$ is usually <u>not</u> representable.

Monoids in $\mathcal{V}\text{-}\mathbf{Cat}[j]$

A monoid in $\mathcal{V}\text{-}\mathbf{Cat}[j]$ comprises

- a $\mathcal{V}$-functor $t : A \to E$
- a $\mathcal{V}$-natural transformation $\eta : j \Rightarrow t$
- a $\mathcal{V}$-form $\mu : E(1, t), E(j, t) \Rightarrow E(1, t)$

satisfying unit and associativity laws.

This looks quite suggestive.

Transposition

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are, by transposition, in bijection with $\mathcal{V}$-forms

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A similar principle applies in the absence of composites.

Relative monads as monoids

Theorem. Let $A \xrightarrow{j} E$ be a $\mathcal{V}$-functor.

There is an isomorphism of categories

$$\mathrm{Mon}(\mathcal{V}\text{-}\mathrm{Cat}[j]) \cong \mathrm{RMnd}(j)$$

Hence, just as every monad is a monoid in a monoidal category, every relative monad is a monoid in a skew-multicategory.

Note that we do not need to impose any assumptions on j.

Equivalent concepts

$\boxed{\text{Relative monads}}$

$\boxed{\text{Monoids in } \mathcal{V}\text{-}\mathbf{Cat}[j]}$

Representability of $\mathcal{V}$-Cat[j]

It is natural to ask when the skew-multicategory $\mathcal{V}$-Cat[j] is representable by a skew-monoidal category.

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Theorem. Suppose that left extensions of $\mathcal{V}$-functors $A \to E$ along $A \xrightarrow{j} E$ are admitted. Then $\mathcal{V}$-Cat[j] is representable.

We thereby recover ACU's characterisation of relative monads as monoids in a SkMC.

Equivalent concepts

Relative monads

Monoids in $\mathcal{V}$-Cat[j]	
SkMulticat	SkMonCat

Formal mw-monads

In 2014 Lack–Street carried out a similar study of skew-monoidal hom-categories to study the extension form of monads. They defined a notion of formal mw-monad, noting an apparent similarity to the relative monads of ACU.

Monoids in $\mathcal{V}$-Dist[j]

We can recover j-relative monads from monoids in $\mathcal{V}$-Dist.

Theorem. There is a pullback

$$\begin{tikzcd}
\mathrm{Mon}(\mathcal{V}\text{-}\mathrm{Cat}[j]) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mon}(\mathcal{V}\text{-}\mathrm{Dist}[j]) \arrow[d] \\
\mathcal{V}\text{-}\mathrm{Cat}(A,E) \arrow[r, hook] & \mathcal{V}\text{-}\mathrm{Dist}(A,E)
\end{tikzcd}$$

Representable formal mw-monads

representable formal mw-monads [ACU10]

formal mw-monads in $\mathcal{V}$-Dist [LS14]

$$
\begin{tikzcd}
\mathrm{RMnd}(j) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Mon}(\mathcal{V}\text{-}\mathrm{Dist}[j]) \arrow[d] \\
\mathcal{V}\text{-}\mathrm{Cat}(A,E) \arrow[r, hook] & \mathcal{V}\text{-}\mathrm{Dist}(A,E)
\end{tikzcd}
$$

Equivalent concepts

Relative monads

Monoids in $\mathcal{V}$-Cat[j]

SkMulticat	SkMonCat

Representable monoids in $\mathcal{V}$-Dist[j]

Relative monads as monads in Dist

In 1975, Diers introduced the notion of j-monad, for $A \xrightarrow{j} E$ a dense, fully faithful functor.

A j-monad is a monoid in Dist(A, A) whose underlying endodistributor is of the form

$$E(j,t) : A \nrightarrow A$$

for some functor $A \xrightarrow{t} E$.

In 2016, Lucyshyn-Wright also studied such monoids, calling such distributors copresheaf-representable.

Distributors to endodistributors

Theorem. There is a skew-multifunctor

$$\mathcal{V}\text{-}\mathrm{Cat}[j] \longrightarrow \mathcal{V}\text{-}\mathrm{Dist}[A, A]$$
$$(A \xrightarrow{t} E) \longmapsto E(j,t) : A \to\!\!\!\!\!+ A$$

which is fully faithful when j is dense.

Hence, when j is dense, we have a pullback

$$\begin{tikzcd} \mathrm{Mon}(\mathcal{V}\text{-}\mathrm{Cat}[j]) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mon}(\mathcal{V}\text{-}\mathrm{Dist}[A,A]) \arrow[d] \\ \mathcal{V}\text{-}\mathrm{Cat}(A, E) \arrow[r, hook, "E(j,-)"'] & \mathcal{V}\text{-}\mathrm{Dist}(A, A) \end{tikzcd}$$

Equivalent concepts

Relative monads

Monoids in $\mathcal{V}$-Cat[j]

SkMulticat	SkMonCat

Representable monoids in $\mathcal{V}$-Dist[j]

j-representable monoids in $\mathcal{V}$-Dist[A, A] *

Monoidality

Theorem. Suppose that left extensions of $\mathcal{V}$-functors $A \longrightarrow E$ along $A \xrightarrow{j} E$ are admitted. Then $\mathcal{V}\text{-}\mathbf{Cat}[j]$ is representable.

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j is f.f.	right-normal

Monoidality

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Furthermore if …	then $\mathcal{V}\text{-}\mathbf{Cat}[j]$ is
j is dense	left-normal
j is f.f.	right-normal
left extensions along j are j-absolute	associative-normal

A characterisation of free cocompletions

Theorem. A $\mathcal{V}$-functor $A \xrightarrow{j} E$ exhibits the free cocompletion of A under a class Φ of weights if and only if

- j is dense and fully faithful
- left extensions along j exist and are j-absolute

Cocontinuous monads

Corollary. Let Φ be a class of weights and let A be a $\mathcal{V}$-category admitting a free Φ-cocompletion $\varphi_A : A \to \Phi A$. Then

$$\mathrm{RMnd}(\varphi_A) \simeq \mathrm{Mnd}_{\Phi}(\Phi A)$$

φ_A-relative monads are equivalent to Φ-cocontinuous $\mathcal{V}$-monads on ΦA.

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φ_A-relative monads are equivalent to Φ-cocontinuous $\mathcal{V}$-monads on ΦA.

Example. Finitary monads on $\mathbf{Set}$ are $(\mathbf{Set}_f \hookrightarrow \mathbf{Set})$-relative monads.

Equivalent concepts

Relative monads

Monoids in $\mathcal{V}$-Cat[j]

SkMulticat	SkMonCat

Representable monoids in $\mathcal{V}$-Dist[j]

j-cocontinuous $\mathcal{V}$-monads *

j-representable monoids in $\mathcal{V}$-Dist[A,A] *

A formal theory of relative monads

For the purpose of this talk, I have worked in the setting of categories enriched in a monoidal category.

However, the results hold much more generally: in any virtual equipment. In particular, this includes $\mathcal{V}$-Cat for any virtual double category $\mathcal{V}$ with restrictions.

Summary

- Relative monads have been rediscovered many times in the last 50 years, albeit in different guises.
- We may understand these fruitfully through distributors and 'skew composition'.
- These viewpoints are useful for proving general theorems about relative monads.

Equivalent concepts

Relative monads

Monoids in $\mathcal{V}$-Cat[j]

SkMulticat	SkMonCat

Representable monoids in $\mathcal{V}$-Dist[j]

j-cocontinuous $\mathcal{V}$-monads *

j-representable monoids in $\mathcal{V}$-Dist[A, A] *

And others...

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